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AN APPROXIMATION TO THE TRUE ABILITY DISTRIBUTION IN THE BINOMIAL ERROR MODEL AND APPLICATIONS

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ABSTRACT

Assuming that the density p of the true ability θ in the binomial test score model is continuous in the closed interval [0,1], a Bernstein polynomial can be used to uniformly approximate p. Then via quadratic programming techniques, least-square estimates may be obtained for the coefficients defining the polynomial. The approximation, in turn will yield estimates for any indices based on the univariate and/or bivariate density function associated with the binomial test score model. Numerical illustrations are provided for the projection of decision reliability and proportion of success in mastery testing.

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1. INTRODUCTION

The binomial error model (Lord and Novick, 1968) has been used extensively in analyses of mental test data. The model is deemed suitable in computer-assisted testing in which each examinee is

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given a random sample of items drawn from a large item universe. When the same test is given to all examinees, the binomial distribution implies that all items share the same difficulty level. There are indications (Keats and Lord, 1962; Duncan, 1974) that several test score distributions based on the same test fit the binomial (or more specifically the beta-binomial) model quite well, especially when similarity of item difficulty holds strictly or nearly. Let x denote the test score obtained from the administration of an n-item test to an examinee with true ability θ (the proportion of items in the universe that he/she knows, or the probability of answering each item correctly). Then the conditional density of x given θ is

$$f(x|\theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}, \quad x = 0,1,\ldots,n.$$

Let $p(\theta)$ be the density of the true ability for a population of examinees. The marginal density of x for this population is given as

$$f(x) = \binom{n}{x} \int_0^1 \theta^x (1 - \theta)^{n-x} p(\theta) \, d\theta.$$

As indicated in Lord and Novick (1968; Chapter 23), the knowledge of $f(x)$ implies the knowledge of the first n moments of the distribution of θ. Any distribution sharing these n moments will yield the same marginal density $f(x)$, hence the solution for $p(\theta)$ given $f(x)$ is not unique. We will seek an approximation for $p(\theta)$ via a polynomial and will show how such approximation is useful in the projection of decision reliability and proportion of successes in mastery testing.

2. A SOLUTION BASED ON THE BERNSTEIN POLYNOMIAL

We shall assume that $p(\theta)$ is continuous in the closed interval [0,1]. Then (Feller, 1966, p. 220) $p(\theta)$ can be uniformly approximated by a Bernstein polynomial of the form

$$B_m(\theta) = \sum_{k=0}^{m} z_k \binom{m}{k} \theta^k (1 - \theta)^{m-k}.$$

Thus given any arbitrarily small and positive ε, there exists an integer m and (m + 1) constants z_i such that $|B_m(\theta) - p(\theta)| < \varepsilon$ for all $\theta \in [0,1]$. We propose to use $B_m(\theta)$ to approximate $p(\theta)$. Procedures will be presented for the determination of the constants $m, z_0, z_1, \ldots, z_m$.

It may first be noted that the z_k constants must be non-negative and satisfy the constraint $\int_0^1 B_m(\theta)\, d\theta = 1$ in order for $B_m(\theta)$ to be a density. Hence

$$\sum_{k=0}^{m} z_k \binom{m}{k} \int_0^1 \theta^k (1 - \theta)^{m-k}\, d\theta = 1$$

or equivalently

$$\sum_{k=0}^{m} z_k = m + 1.$$

The Bernstein approximated value for the marginal density of x is now given as

$$f_B(x) = \binom{n}{x} \sum_{k=0}^{m} z_k \binom{m}{k} J(n + m;\ x + k)$$

where

$$J(n + m;\ x + k) = \int_0^1 \theta^{x+k} (1 - \theta)^{n+m-(x+k)} d\theta.$$

The J integrals may be computed inductively by noting that

$$J(p;0) = 1/(p + 1)$$

and

$$J(p;y + 1) = (y + 1)\, J(p;y)/(p - y).$$

Now let

$$c(k,x) = \binom{n}{x}\binom{m}{k} J(x + k)$$

and

$$\alpha(k,x) = c(k,x) - c(0,x).$$

Then the approximated marginal density of x becomes

$$f_B(x) = \sum_{k=1}^{m} \alpha(k,x)\, z_k + (m + 1)\, c(0,x)$$

where the z_k, $k = 1, 2, \ldots, m$ are nonnegative and sum up to no more than m + 1.

To determine the constants m, z_1, z_2,..., z_m, we focus on the least-square criterion with the weight function w(x)

$$H(z_1, z_2, \ldots, z_m; m) = \sum_{x=0}^{n} w(x) [f_B(x) - f(x)]^2. \quad (1)$$

In other words, we will seek these constants in such a way that the H criterion is minimized. This may be done by first considering m as fixed and computing the z constants along with the minimum H_m of the criterion H. This process will be repeated many times starting with m = 0 [$p(\theta)$ and $f_B(x)$ are constant], 1, 2, etc. until an integer m can be located at which H_m is minimized. Following are the details for the algorithm.

2.1 Minimizing H at Each Integer m. Let

$$\beta(x) = (m + 1)\, c(0,x) - f(x).$$

Then (1) becomes

$$H = \sum_{x=0}^{n} [w(x) \sum_{k=1}^{m} \alpha(k,x) z_k + \beta(x)]^2. \quad (2)$$

At each given integer m, the nonnegative z_1, z_2,..., z_m may be obtained by minimizing H under the constraint $\Sigma z_k \leq m + 1$. Since H is continuous and the z's are located in a closed region, the solution for z always exists. To obtain such solution, standard routines for quadratic programming may be called upon. In this paper, Algorithm 431 (Ravindran, 1972) was used.

To enter into Algorithm 431, we note that the criterion H of (2) may be written as

$$H = Z'DZ + 2BZ + C.$$

In this formula, Z is the vector $(z_1, z_2, \ldots, z_m)'$, $D = (d_{kk'})$ is the matrix defined by

$$d_{kk} = \sum_{x=0}^{n} w(x) [\alpha(k,x)]^2$$

$$d_{kk'} = \sum_{x=0}^{n} w(x)\, \alpha(k,x)\, \alpha(k',x)$$

and $B = (b_k)$ is a vector with components

$$b_k = \sum_{x=0}^{n} w(x)\, \alpha(k,w)\, \beta(x).$$

The remaining quantity C is the constant

$$C = \sum_{x=0}^{n} w(x)\, [\beta(x)]^2.$$

2.2 Searching the Least Square Solution. We note that when $m = 0$, the minimum value H_0 of H is simply

$$H_0 = \sum_{x=0}^{n} w(x)\, [f(x) - \bar{f}]^2$$

where

$$\bar{f} = \Sigma w(x)\, f(x)/\Sigma w(x).$$

As for other m values, the minimum may be deduced from the quadratic programming. Thus the least square solution for the Bernstein polynomial may be obtained by computing H_0, H_1, H_2,... for several consecutive values of m, and locating the value of m at which H_m is the smallest. Since the criterion for minimization H is non-negative, all computations shall stop whenever $H_m = 0$. In other situations, a tolerance difference between H_m and H_{m-1} might have to be set up in order to end the approximation process.

3. NUMERICAL ILLUSTRATION

To illustrate the computational algorithm described in the previous section, three score frequency distributions based on $n = 10$ test items are used. For Data Set 1, almost all frequencies are concentrated at the upper end of the score range. Data Set 2 is slightly asymmetric and Data Set 3 has two modes, one near each end of the score range. Details regarding these data sets are presented in Table 1.

It appears from Table I that the goodness of fit via the

Bernstein polynomial improves when the degree of the polynomial increases. For unimodal distributions, the algorithm tends to put all the weights at only a few terms which correspond to some

TABLE 1

Observed and Fitted Frequency Distributions for Three Data Sets

Test Score	Data Set 1		Data Set 2		Data Set 3	
	Observed	Fitted	Observed	Fitted	Observed	Fitted
0	0	.00	0	.06	4	6.09
1	0	.00	0	.37	10	10.28
2	0	.01	1	1.26	15	10.16
3	0	.07	3	3.07	2	8.68
4	1	.23	6	5.97	6	9.16
5	1	.69	10	9.66	10	12.64
6	3	1.82	13	13.28	20	17.49
7	5	4.42	16	15.47	25	20.22
8	8	9.93	15	14.88	15	17.89
9	15	20.96	11	11.00	10	10.89
10	47	41.91	5	4.97	4	3.50
Degree of the Bernstein polynomial:		10		10		24
Minimum H_m:		.0106		.0001		.0052
The positive z constants:						
		$z_{10} = 11.0000$		$z_7 = 9.8912$		$z_4 = 6.2830$
				$z_8 = 1.1088$		$z_5 = 1.3349$
						$z_{17} = 14.4010$
						$z_{18} = 3.9830$

consecutive z_i values. On the other hand, for a bimodal distribution such as Data Set 3, the algorithm puts the total weight on two blocks, each being formed by some consecutive z_i values.

4. PROJECTION OF DECISION RELIABILITY

Consider now two equivalent tests X and Y, each with n items. If the test score distributions are binomial, then the bivariate density is given as

$$f(x,y) = \binom{n}{x}\binom{n}{y} \int_0^1 \theta^{x+y} (1-\theta)^{2n-(x+y)} p(\theta)\, d\theta.$$

Let the density p be approximated from the data collected with one test as

$$B_m(\theta) = \sum_{k=0}^{m} z_k \binom{m}{k} \theta^k (1-\theta)^{m-k}.$$

Then f(x,y) will be given by the expression

$$f_B(x,y) = \binom{n}{x}\binom{n}{y} \sum_{k=0}^{m} z_k \binom{m}{k} J(2n+m;\ x+y+k)$$

where the function J is defined as previously in Section 2. The expressions for $f_B(x)$ and $f_B(x,y)$ may now be used to project practically all agreement indices for decisions in mastery testing. Let the examinees now be classified in k categories A_i defined by $A_i = \{x; c_{i-1} \leq x < c_i\}$ where $c_0 = 0$ and $c_k = n + 1$. For binary classifications $k = 2$. In this case c_1 is usually referred to as the cutoff (mastery) score. The raw agreement index

$$P = \sum_{i=1}^{m} P\,[(X,Y) \in A_i \times A_i]$$

can be computed by the formula

$$P = \sum_{i=1}^{k} \Big[\sum_{c_{i-1} \leq x,y < c_i} f_B(x,y) \Big].$$

On the other hand, the corrected-for-chance kappa index is given as $\kappa = (P - P_c)/(1 - P_c)$ where

$$P_c = \sum_{i=1}^{k} \Big[\sum_{c_{i-1} \leq x < c_i} f_B(x) \Big]^2.$$

4.1 Numerical Example. Consider the case where $n = 5$, $m = 4$ and $z_0 = 1.0$, $z_1 = 1.5$, $z_2 = 2.0$, $z_3 = 0$ and $z_4 = .5$. The Bernstein polynomial generates the marginal frequency density of .20040, .21230, .20040, .16865, .12698 and .09127 at the test scores of 0, 1, 2, 3, 4, and 5. For the binary classifications with cutoff score 4, the raw agreement index is .8197 and the kappa index is .4716.

5. PROJECTION OF TEST SCORE DISTRIBUTIONS FOR LENGTHENED TESTS

There are situations in which a test needs to be lengthened in order to accomodate new conditions and data are available for the short version of the test. If the binomial model holds, then it is possible to project the test score distribution for a lengthened test, assuming that the ability distribution of the examinees remains unchanged. From the data for the short form, it may be possible to approximate the true ability distribution via the Bernstein polynomial

$$B_m(\theta) = \sum_{k=0}^{m} z_k \binom{m}{k} \theta^k (1 - \theta)^{m-k}.$$

For a lengthened test consisting of ℓ items, the projected density function for the test score is given as

$$f(x) = \binom{\ell}{x} \int_0^1 \theta^x (1 - \theta)^{\ell - x} p(\theta)\, d\theta$$

$$= \binom{\ell}{x} \sum_{k=0}^{m} z_k \binom{m}{k} J(\ell + m, x + k).$$

5.1 Numerical Example. Consider the case where the fitting via a 4th degree Bernstein polynomial ($m = 4$) yields the constants $z_0 = 1.0$, $z_1 = 1.5$, $z_2 = 2.0$, $z_3 = 0$ and $z_4 = .5$. For a test with $\ell = 10$ items, the projected density is .10406, .11372, .11888, .11905, .11422, .10489, .09207, .07726, .06244 and .05012 at the test scores of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

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